Engineering Notes

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Computation of Optimal Feedback Strategies for Interception in a Horizontal Plane

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Introduction

IN Refs. 1-4, extremals were derived for optimal minimum-time interception of one aircraft by another in a horizontal plane. These analyses used the necessary conditions of differential game theory[5] to compute fields of open-loop extremals called extremal trajectory maps (ETM). The aircraft models used in these analyses employed variable speeds and turn rates, realistic aerodynamic and propulsive force models, and realistic constraints. Both a passive and an optimally evading target were considered.

In this Note, the ETMs of Refs. 1-4 are used to derive feedback solutions to the minimum-time interception problem in a horizontal plane. The feedback solutions are obtained by constructing isochrones (constant minimum-time loci). The construction of feedback solutions is complicated by the existence of singular surfaces, such as Barriers and dispersal surfaces.[5] As before, both passive and actively evading targets are considered. The numerical examples use the F-4C aircraft model of Ref. 6.

A feedback solution to the three-dimensional interception problem is essential for eventual automatic onboard flight path management of air combat aircraft. Because this problem involves six state variables and is highly nonlinear, it has not been possible to derive a feedback solution for the three-dimensional case.[7] The two-dimensional analysis described in this Note is a necessary step toward the solution of the three-dimensional problem.

The Feedback Solution

A feedback solution to the interception or pursuit-evasion problem must give the control strategies and the time to capture at any point in the state-space. One way of presenting the solution is to draw the boundaries of the regions within which the strategies are constant. Where a control varies continuously over a range, as does the bank here, the surfaces at which the control attains specified values are mapped. Since these boundaries and surfaces are in a four- or five-dimensional state-space, they can be pictured only by cross sectioning. They are sectioned by keeping constant the initial Mach numbers of the interceptor and target (or of the pursuer and evader) and the initial relative heading. The cross sections are then plotted relative to the interceptor/pursuer and become curves in the plane of the encounter.

The control level surfaces, Barrier points, and dispersal points are all located by drawing cross sections of the isochrones. A control level surface links the points on the isochrone at which the controls take on a specified value. Dispersal points are located at the intersections of different branches of the same isochrone. Where the isochrone cross sections are not closed, their end points are on a Barrier cross section. Such a section separates two families of isochrones. From the ETMs of the interceptor and target, the target's initial position relative to the interceptor is calculated for each pair of extremals, giving a candidate point on an isochrone section.

The interceptor and target ETMs consist of extremals generated for the same time-to-go, with the terminal headings β_{If} and β_{Tf} as parameters. For each extremal, the initial Mach numbers are M_{I0} and M_{T0}, and the initial heading angles measured relative to the terminal line-of-sight are β_{I0} and β_{T0}. The isochrone section being constructed has M_{I0}, M_{T0}, and β_0 specified, where β_0 is the initial relative heading

$$\beta_0 = \beta_{I0} - \beta_{T0} \tag{1}$$

For any given pair of terminal headings β_{If} and β_{Tf}, the terminal Mach numbers M_{If} and M_{Tf} have already been iterated to match the starting Mach numbers M_{I0} and M_{T0}. For matching the relative heading, β_{If} is taken as a fixed parameter, and β_{Tf} is determined by searching in the target's ETM.

Different values of β_{If} give other points on the isochrone section. The values of β_{If} are selected such that all significant points on the section are mapped. These include the zero-, half-, and full-bank points (the bank at the start is zero, half, and full, respectively) and any point where the bank or throttle switches. For any given section, some of the above points may not appear because the reachability condition is not satisfied or because they fall beyond a dispersal point. All these points depend on only one vehicle's terminal heading, either β_{If} or β_{Tf}. Once marked in an ETM, mapping them on any isochrone section requires only a search of the other vehicle's ETM. A dispersal point occurs if, for either of the vehicles, two extremals with different values of the terminal heading pass through the same point in relative space.

For a nonevading target whose motion is known in advance, only the orientation of its terminal velocity vector relative to the terminal line-of-sight changes. The target's ETM consists of the same path rotated through different terminal headings. For any given value of τ_f, the target's turn angle is constant and independent of the terminal heading. The latter is computed such that Eq. (1) is met for the β_0 value specified for the section. Thus, unlike in the case of an evading target, searching of the other vehicle's ETM is unnecessary.

In Fig. 1b, the dispersal line is also shown. For initial target positions along this line, the interceptor can initially choose to bank left or right and still obtain the same payoff. In Fig. 1c, the dispersal line is the negative y axis. For initial target positions behind the interceptor but within the zero-throttle

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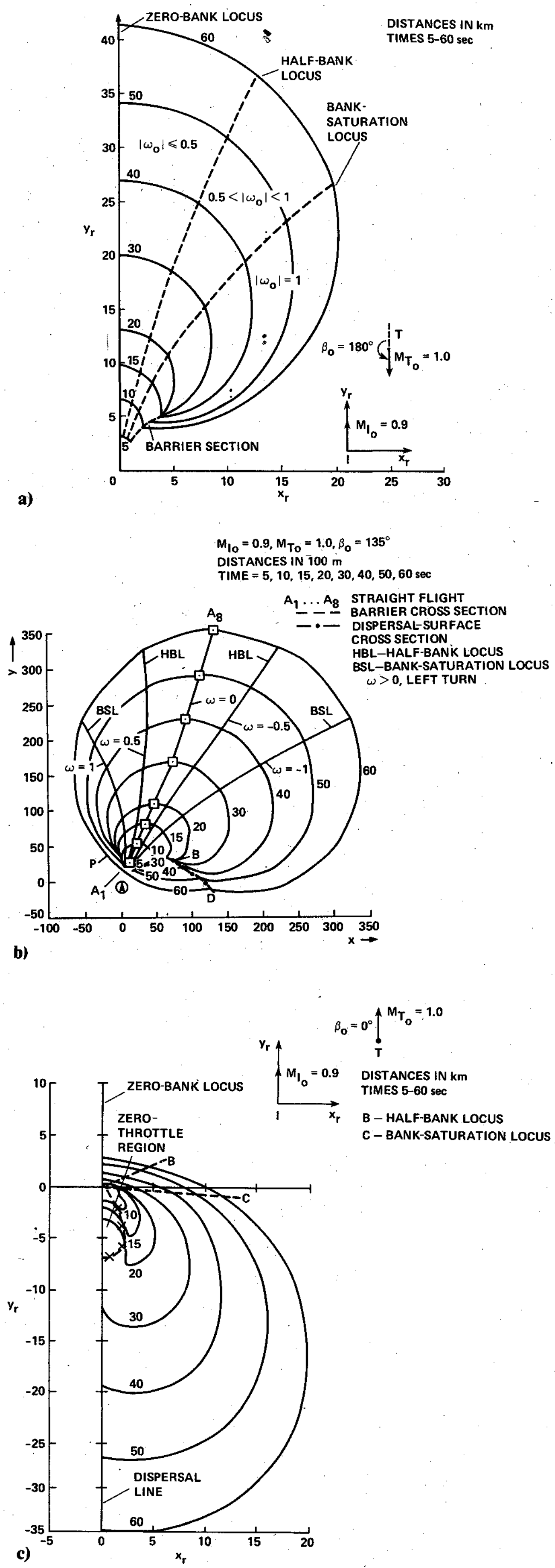

Fig. 1 Sections of the feedback solution for a passive target: a) initial relative heading = 180 deg; b) initial relative heading = 135 deg; initial relative heading = 0 deg.

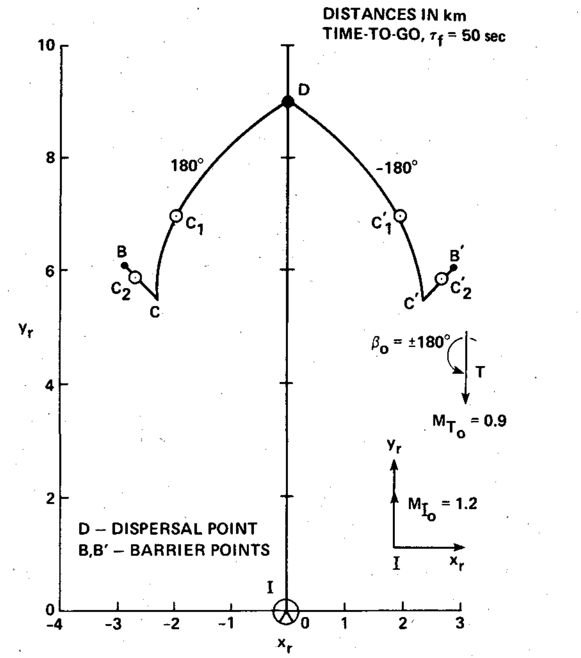

Fig. 2 An isochrone section for an optimally evading target, initial relative heading = 180 deg.

region, the interceptor employs zero throttle and full bank, together with bank switches to decelerate and let the target overtake it. However, if the target starts far behind it (a distance greater than 10 km), the interceptor has time to turn toward the target and accelerate. In either case, the interceptor can choose to turn left or right initially.

An isochrone section for an actively evading target is shown in Fig. 2. The initial speeds of the interceptor and target are Mach 1.2 and 0.9, respectively. The same aircraft model is used for both. The capture radius is 316 m. The section is drawn for a time-to-go of 50 s and an initial relative heading $\beta_0 = 180$ deg. D (Fig. 2), is a dispersal point because the target can choose to turn left or right and still be captured in the same time. Starting from D, the target essentially tries to turn almost completely around and flee. The interceptor turns around 30 deg to align itself with the anticipated target heading and accelerates. From D to B (Fig. 2), the target's heading change decreases while that of the interceptor increases. At B, the target turns a little away from the interceptor and accelerates. The interceptor turns around 150 deg and gives chase. Beyond B, the interceptor cannot capture the target because the latter can accelerate to maximum speed before the former can close in. Thus, B is a point on a closed Barrier section.

Examples

A target flying in a straight line at a constant speed of Mach 1.0 and at an altitude of 6.1 km is to be intercepted in minimum time by a supersonic aircraft (a version of the F-4C)[5] flying at the same altitude at an initial Mach number of 0.9. A capture radius of 316 m is specified. Isochrone sections for the capture times of 5-60 s are shown in Fig. 1 for different initial relative target headings. The throttle is full for all the sections in Fig. 1a and for all but a very small part of the 5 s locus in Fig. 1b. In Fig. 1c, the region within which all extremals start with zero throttle and then switch to full is shown. The zero-bank, half-bank, and bank-saturation loci are also shown in all three figures; these link the initial target positions for which the interceptor flies straight ahead to intercept the target, for which the interceptor starts out with its bank angle at half the maximal value, and with the starting bank just saturated, respectively. In Figs. 1a and 1b, the isochrone sections for τ_f varying 5-20 s terminate on a Barrier section, because the reachability condition[3] becomes an equality. However, the interceptor has a maximum speed capability that is higher than the target speed, and so it can accelerate and eventually capture the target. This is reflected by the discontinuity in the time-to-capture across the Barrier section.

Conclusion

Sections of the feedback solution for interception in a horizontal plane have been mapped by constructing isochrones from open-loop extremals of the interceptor and the target. Both a passive and an optimally evading target were considered. Construction of the feedback solution requires little additional computational effort over that needed to compute the open-loop extremals, and the feedback solution should be capable of onboard implementation.

The aircraft dynamic models employed in the analysis are realistic in terms of aerodynamic forces and constraints, but the restriction to a horizontal plane makes the feedback solutions of limited practical value. The horizontal plane analysis, however, is a necessary step toward developing feedback solutions for the three-dimensional problem.

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An Elementary Proof of the Optimality of Hohmann Transfers

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Introduction

A GRADIENT method is used to prove analytically that the Hohmann transfer is the optimal two-impulse transfer between coplanar circular orbits in a central force field with a Newtonian attraction. The proof is elementary in that only well-known properties of conics are assumed. By an extension of the method, it is shown that a Hohmann type transfer is the optimal two-impulse transfer between coplanar orbits, one of which is circular and the other elliptical.

The Hohmann transfer was proposed in Ref. 1. Barrar[2] gives analytic proofs of several results on optimal transfer between coplanar orbits. Marec[3] gives a proof by graphical construction of the optimality of the Hohmann transfer between coplanar circular orbits. Lawden[4] uses extensively the calculus of variations to find solutions to these transfer problems.

The gradient method is distinct from those above. It differs from a critical point method in that it shows clearly that the characteristic velocity attains a minimum on the boundary of the region where the transfer orbits lie.

The gradient method can be applied to show that the bielliptic transfer is the optimal transfer among all three-impulse transfers between coplanar circular orbits. Furthermore, the method can also be used to show that the multielliptic transfer is the optimal transfer among all N-impulse transfers between coplanar circular orbits for $N>3$.

Optimality of the Hohmann Transfer Between Coplanar Circular Orbits

The transfer orbit is a conic that is represented by $p/r=1+e\cos\theta$ where $e\geq 0$ is the eccentricity, $2p$ is the length of the latus rectum and r is the distance from the focus. Units may be chosen so that the gravitational constant is equal to 1. An orbit with parameters (p,e) has associated with it an energy $(e^2-1)/2p$ and an angular momentum $p^{1/2}$. A change in velocity from a circular orbit of radius R to a conic with parameters (p,e) has a magnitude ΔV given by

$$(\Delta V)^2=v^2+v_c^2-2v_c v_\theta$$

where

$$v^2=2/R+(e^2-1)/p,\quad v_c^2=1/R\quad\text{and}\quad v_\theta=p^{1/2}/R$$

By introducing new variables $x=p^{-1/2}$ and $y=ep^{-1/2}$ the magnitude ΔV can be written as

$$(\Delta V)^2=3/R+y^2-x^2-2/R^{3/2}x$$

Let the radii of the circular orbits be R_1 and R_2 with $R_1<R_2$. Let $\mathcal{R}$ denote the region

$$\{x>0,\ y>0\ \text{and}\ x^2-xy\leq R_2^{-1}<R_1^{-1}\leq x^2+xy\}$$

The requirement that $x^2+xy\geq R_1^{-1}$ be satisfied means that the periapsis of the transfer conic lies within distance R_1 of the focus; the requirement that $x^2-xy\leq R_2^{-1}$ be satisfied means that the apoapsis lies at a distance from the focus not less than R_2. The region $\mathcal{R}$ has a corner (x_0,y_0) at the intersection of the curves defined by $x^2-xy=R_2^{-1}$ and $x^2+xy=R_1^{-1}$. At the corner the values of x_0 and y_0 are

$$x_0=[(R_1+R_2)/2R_1R_2]^{1/2}$$

$$\text{and}\quad y_0=(R_2-R_1)/[2R_1R_2(R_1+R_2)]^{1/2}$$

Figure 1 shows the region $\mathcal{R}$. The diagonal divides $\mathcal{R}$ into two subregions of ellipses (I), $y<x$, and hyperbolas (II), $y>x$. The diagonal represents parabolas for which $e=1$.

We denote the characteristic velocity $\Delta V_1+\Delta V_2$ of an orbital transfer by V_{CH}. The partial derivative $\partial V_{CH}/\partial y$ is given by $y[(\Delta V_1)^{-1}+(\Delta V_2)^{-1}]$ and is positive throughout $\mathcal{R}$. The gradient $(\partial V_{CH}/\partial x,\ \partial V_{CH}/\partial y)$ of the characteristic velocity is normal to the level curves $V_{CH}=$constant. The negative gradient of V_{CH} points in the direction of the maximum decrease of V_{CH}. At any interior point of $\mathcal{R}$ the

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